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The K-divisibility constant for couples of Banach lattices

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Abstract

It is shown that the Brudnyi–Krugljak K-divisibility constant for an arbitrary couple of Banach lattices on the same underlying measure space is bounded above by 4.

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0. Introduction

The Brudnyi–Krugljak K-divisibility theorem [3,4, p. 325] is one of the most important and useful results in real interpolation theory, and potentially also has interesting applications beyond that theory. Let us recall its formulation:

Theorem 0.1. *Let $\vec{A} = (A_0, A_1)$ be a Banach couple. There exists a constant C, depending only on $\vec{A}$, which has the following property.*

Suppose that

$$K(t, a; \vec{A}) \leqslant \sum_{n=1}^{\infty} \phi_n(t) \quad \textit{for all } t > 0,$$

where a is an arbitrary element of $A_0 + A_1$ and the functions ϕ_n are each positive and concave on $(0, \infty)$ and $\sum_{n=1}^{\infty} \phi_n(1) < \infty$. Then there exist elements $a_n \in A_0 + A_1$ such

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that $a=\sum_{n=1}^{\infty} a_n$ (*where this series converges in* A_0+A_1 *norm*) *and*

$$K(t, a_n; \vec{A}) \leqslant C\phi_n(t) \quad \textit{for all } t>0 \textit{ and each } n \in \mathbb{N}.$$

For more details about this theorem and its applications we refer to [4] and also to remarks in the introductions of [5,6].

It is customary to use the notation $\gamma(\vec{A})$ for the *K-divisibility constant for* $\vec{A}$, i.e. the infimum of all numbers C having the property stated in Theorem 0.1. It is known (cf. [6]) that

$$1 \leqslant \gamma(\vec{A}) \leqslant 3+2\sqrt{2} \tag{1}$$

for every Banach couple $\vec{A}$. For more information about the value of $\gamma(\vec{A})$ for various particular couples we refer to [7] (in particular p. 29) and the papers cited in [7]. We also refer to a forthcoming paper by Y. Ameur and the author.

It is known (see [7, Section 7]) that estimate (1) can sometimes be sharpened for a whole class of couples $\vec{A}=(A_0, A_1)$, when the spaces A_0 and A_1 have additional structure. The case considered in [7] is when A_0 and A_1 are both real Banach spaces of (equivalence classes of) measurable real valued functions on the same measure space $(\Omega, \mathscr{S}, \mu)$ and they are both Banach lattices. More specifically, it is assumed that, for $j=0,1$,

$$\left\{ \begin{array}{l} \text{if } f \text{ and } g \text{ are measurable functions on } \Omega \text{ which satisfy } |g(\omega)| \leqslant |f(\omega)| \\ \text{for a.e. } \omega \in \Omega \text{ and if } f \in A_j, \text{ then } g \in A_j \text{ with } \|g\|_{A_j} \leqslant \|f\|_{A_j}. \end{array} \right\} \tag{2}$$

In fact all results in [7] can also be formulated and proved in essentially the same way, in the complex case, i.e., where A_0 and A_1 are complexified Banach lattices on $(\Omega, \mathscr{S}, \mu)$, namely, complex Banach spaces of complex valued measurable functions on Ω which satisfy (2).

We shall use the terminology *lattice couple* for a Banach couple $\vec{A}$ which is a couple of either real or complexified Banach lattices of the kinds described in the preceding paragraph. It is shown in Section 7 of [7], that every lattice couple $\vec{A}$ satisfies

$$\gamma(\vec{A}) \leqslant 4\lambda(\vec{A}), \tag{3}$$

where $\lambda(\vec{A})$ is (cf. [7, Remark 7.4, p. 53]) the infimum of those numbers $\lambda \geqslant 1$ for which $\vec{A}$ has the property of λ-*monotonicity* defined in [7, Definition 1.4, p. 30]. It is shown in [7] that $\lambda(\vec{A})=1$ for many particular lattice couples, but also that there are examples where $\lambda(\vec{A})>1$.

In this note we refine earlier alternative proofs of Theorem 0.1 which were given in [5–7]. As in [7], we only consider the case where $\vec{A}$ is a lattice couple. Our new proof of Theorem 0.1 gives a sharpening of (3) for lattice couples, namely

$$\gamma(\vec{A}) \leqslant 4. \tag{4}$$

It is interesting to note that Brudnyi and Krugljak [4, p. 492] have claimed that there are sound reasons to believe that estimate (4) holds for all Banach couples $\vec{A}$.

Our proof of (4) will be given in the next and final section of this note.

We will assume familiarity with the basic notions of the real method of interpolation, as presented, e.g. in [1,2] or [4]. We will use the notation $A_j^{\sim}$ for the *Gagliardo completion* of A_j, $j = 0, 1$, i.e. the set of elements a of $A_0 + A_1$ which are limits in $A_0 + A_1$ norm of bounded sequences in A_j or, equivalently, for which the norm $||a||_{A_j^{\sim}} = \sup_{t>0} K(t, a; \vec{A})/t^j$ is finite. Because of the monotonicity of $K(t, a; \vec{A})$ and of $K(t, a; \vec{A})/t$ we also have $||a||_{A_0^{\sim}} = \lim_{t\to\infty} K(t, a; \vec{A})$ and $||a||_{A_1^{\sim}} = \lim_{t\to 0} K(t, a; \vec{A})/t$. Of course

$$||a||_{A_j^{\sim}} \leqslant ||a||_{A_j} \quad \text{for each } a \in A_j. \tag{5}$$

We will also assume some familiarity with the elementary properties of Banach lattices of measurable functions. In particular, we will need the following easily proved fact (cf. [12, Exercise 64.1, p. 446]).

Proposition 0.2. *Let A be a Banach lattice of measurable functions on the measure space $(\Omega, \mathscr{S}, \mu)$ and suppose that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in A satisfies $\sum_{n=1}^{\infty} ||f_n||_A < \infty$. Then there exists a function $f \in A$ such that*

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} f_n - f \right\|_A = 0$$

and also $f(\omega) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(\omega)$ for μ-a.e. $\omega \in \Omega$.

1. The proof itself

We will use many of the features of the proofs given in [5–7]. In particular, our main step will be to prove the following version of the so-called "strong fundamental lemma":

Theorem 1.1. *Let $\vec{A} = (A_0, A_1)$ be a lattice couple and let $\vec{A}^{\sim}$ denote the couple $(A_0^{\sim}, A_1^{\sim})$ where $A_j^{\sim}$ is the Gagliardo completion of A_j in $A_0 + A_1$, $j = 0, 1$. Let $a \in A_0 + A_1$. Then for each $\varepsilon > 0$ there exists a sequence of elements $\{u_{n,\varepsilon}\}_{n\in\mathbb{Z}} = \{u_n\}_{n\in\mathbb{Z}}$ in $A_0 + A_1$ such that:*

$u_n \in A_0 \cap A_1$ for all but at most two values of n,

$\sum_{n=-\infty}^{\infty} u_n = a$, (convergence in $A_0 + A_1$ norm), and

$$\sum_{n=-\infty}^{\infty} \min\{||u_n||_{A_0^{\sim}}, t||u_n||_{A_1^{\sim}}\} \leqslant 4(1+\varepsilon)K(t, a; \vec{A}) \quad \textit{for all } t > 0. \tag{6}$$

(In the preceding estimate we set $||u_n||_{A_j^{\sim}} = \infty$ if $u_n \notin A_j^{\sim}$.)

Once we have established Theorem 1.1, we can deduce Theorem 0.1 with estimate (4) in almost exactly the same way as Theorem 1 of [5] is deduced from Theorem 4 of [5]. We will not reproduce the argument for doing this [5, pp. 54–55] here since there is only one change, an obvious one; the constant 8 appearing in [5] has to be replaced here by 4.

Remark 1.2. The argument to which we have just referred does not use the fact that $\vec{A}$ is a lattice couple. In fact, it seems reasonable to conjecture that it also works in the reverse direction, i.e., that any Banach couple $\vec{A}$ satisfies the conclusion of Theorem 0.1 with (4) if and only if it satisfies the conclusion of Theorem 1.1. This equivalence is known to hold whenever $A_j^\sim = A_j$ isometrically for $j = 0, 1$ or when, in the conditions formulated in Theorem 1.1, the assertion $u_n \in A_0 \cap A_1$ is replaced by $u_n \in A_0^\sim \cap A_1^\sim$. For details we refer to Remarks 1.34 and 1.36 and Proposition 1.40 of [8].

We now begin the proof of Theorem 1.1. Some steps of our proof are identical, or almost identical to certain parts of the proof presented in Section 2 of [6, pp. 73–77]. So we will sometimes refer to that paper, rather than reproducing those parts here. On the other hand, other steps in the proof given in [6] were explained rather briefly, sometimes appealing to intuitive geometric arguments. Here we will offer more explicit and detailed explanations of those steps, for whoever may find them helpful.

Let us fix an element $a \in A_0 + A_1$. As in [6], we use the abbreviated notation $K(t)$ for the K-functional $K(t, a; \vec{A})$ of a. In fact, except for one change, we will use exactly the same notation throughout as in [6]. This change (or pair of changes) is that we have permuted the definitions from [6] of the quantities y_∞ and $y_{-\infty}$ (see (13) and (15)). Our new usage matches more naturally with the notation for the sequence $\{y_n\}$ defined later on in the proof.

There are two special cases where the proof of Theorem 1.1 is immediate and it is convenient to dispose of them now. These are when, for some constant $c > 0$, we have either

$$K(t) = c \quad \text{for all } t > 0, \quad \text{or} \quad K(t) = ct \quad \text{for all } t > 0. \tag{7}$$

In the first case we have that $||a||_{A_0^\sim} = c$ and in the second $||a||_{A_1^\sim} = c$. In each of these cases we obviously obtain (6) (and with a rather better constant) by simply choosing $u_0 = a$ and $u_n = 0$ for all $n \neq 0$.

We recall the definition of the *Gagliardo diagram* of a, i.e. the set

$$\Gamma(a) = \{(x_0, x_1) \in \mathbb{R}^2 \mid \exists a_j \in A_j \text{ s.t. } ||a_j||_{A_j} \leqslant x_j, \ j = 0, 1, \ a = a_0 + a_1\}$$

(cf. [2, p. 39; 9, 10]). Two obvious but important properties of this set are that it is convex and that it is "monotonic" in the sense that $(x, y) \in \Gamma(a)$ whenever $x \geqslant x'$ and $y \geqslant y'$ for some point $(x', y') \in \Gamma(a)$. The set $\overline{\Gamma(a)}$ (the closure of $\Gamma(a)$) also has both these properties. Yet another obvious property is that

$$a \neq 0 \text{ if and only if } \inf\{s > 0 \colon (s, s) \in \Gamma(a)\} > 0. \tag{8}$$

The set $\Gamma(a)$ is of course closely related to the K-functional of a, and in particular it is clear (cf. e.g. [2]) that

$$\left\{\begin{array}{l}\text{For each } t>0, \text{ the line } x+ty=K(t) \text{ has non empty intersection}\\ \text{with } \overline{\Gamma(a)} \text{ and is disjoint from the interior of } \Gamma(a).\end{array}\right\} \tag{9}$$

It is easy to check that the two special cases listed in (7) are equivalent, respectively, to $\overline{\Gamma(a)}$ having one of the special forms

$$\overline{\Gamma(a)} = \{(x,y)\colon\ x \geqslant c, y \geqslant 0\} \quad \text{or} \tag{10}$$

$$\overline{\Gamma(a)} = \{(x,y)\colon\ x \geqslant 0, y \geqslant c\} \quad \text{for some } c>0.$$

We will make substantial use of the subset $D(a)$ of the boundary $\partial\Gamma(a)$ of the Gagliardo diagram of a which does not meet the coordinate axes in $\mathbb{R}^2$, i.e.

$$D(a) = \partial\Gamma(a) \cap \{(x_0, x_1) \in \mathbb{R}^2\colon\ x_j > 0,\ j = 0, 1\}.$$

As well as excluding cases (10), we can and will assume from this point onwards that $D(a)$ is non-empty. We can do this because, as follows immediately from (8) (and as remarked in [7] and overlooked in [6]), $D(a)$ is empty if and only if $a = 0$, in which case Theorem 1.1 is of course a triviality.

The following two "claims" establish several properties of the sets $D(a)$ and $\partial\Gamma(a)$ which we will need to use later. Most of these properties are established explicitly or implicitly in [6] but, as mentioned above, we feel it may perhaps be helpful to formulate and prove them in a more detailed way.

Claim 1.3. *There exist a point $(x_0, y_0) \in D(a)$ and two non-increasing continuous convex functions $\phi : [x_0, \infty) \to [0, y_0]$ and $\psi : [y_0, \infty) \to [0, x_0]$ such that $\phi(x_0) = y_0$ and $\psi(y_0) = x_0$ and*

$$\partial\Gamma(a) = \partial\Gamma(a)_- \cup \partial\Gamma(a)_+, \tag{11}$$

$$\textit{where } \partial\Gamma(a)_- = \{(\psi(y), y)\colon\ y \in [y_0, \infty)\}$$

$$\textit{and } \partial\Gamma(a)_+ = \{(x, \phi(x))\colon\ x \in [x_0, \infty)\}.$$

Furthermore,

$$D(a) = D(a)_- \cup D(a)_+, \tag{12}$$

$$\textit{where } D(a)_- = \{(\psi(y), y)\colon\ y \in [y_0, y_{-\infty})\}$$

$$\textit{and } D(a)_+ = \{(x, \phi(x))\colon\ x \in [x_0, x_\infty)\},$$

where

$$x_\infty := \sup\{x \mid (x,y) \in D(a)\} \quad \textit{and} \quad y_{-\infty} := \sup\{y \mid (x,y) \in D(a)\}. \tag{13}$$

Proof. Let (x_0, y_0) be the point of $\partial\Gamma(a)$ which satisfies $\sqrt{x_0^2 + y_0^2} = \inf\{\sqrt{x^2 + y^2}\colon (x, y) \in \partial\Gamma(a)\}$. If $y_0 = 0$ then it follows immediately from the "monotonicity" and convexity of $\overline{\Gamma(a)}$ that $(x, y) \notin \overline{\Gamma(a)}$ whenever $x < x_0$. This in turn implies that $\overline{\Gamma(a)} = \{(x, y)\colon x \geqslant x_0, y \geqslant 0\}$. Similarly, we have $x_0 = 0$ if and only if $\overline{\Gamma(a)} = \{(x, y)\colon x \geqslant 0, y \geqslant y_0\}$. Since we have excluded the cases where $\overline{\Gamma(a)}$ is of the forms listed in (10), we deduce that x_0 and y_0 are both strictly positive, i.e. $(x_0, y_0) \in D(a)$.

For each constant $u \geqslant x_0$ it is clear that the line $x = u$ intersects $\Gamma(a)$. Consequently, the function ϕ defined on $[x_0, \infty)$ by $\phi(u) := \inf\{s \in \mathbb{R}, (u, s) \in \overline{\Gamma(a)}\}$ is finite, non-negative, and satisfies $(x, \phi(x)) \in \partial\Gamma(a)$ for each $x \in [x_0, \infty)$. Obviously ϕ is non-increasing (because of the "monotonicity" of $\overline{\Gamma(a)}$). Since $\overline{\Gamma(a)}$ is a convex set, we also have that ϕ is a convex function. The definition of (x_0, y_0) ensures that $\phi(x_0) = y_0$.

In view of its convexity on $[x_0, \infty)$, the function ϕ is continuous on (x_0, ∞). It follows that (x, y) is an interior point of $\Gamma(a)$ whenever $x > x_0$ and $y > \phi(x)$. Consequently, the set $\{(x, y) \in \partial\Gamma(a)\colon x > x_0\}$ coincides with the graph $\{(x, \phi(x))\colon x > x_0\}$. Similarly, the set $\{(x, y) \in D(a)\colon x > x_0\}$ coincides with $\{(x, \phi(x))\colon x_0 < x < c\}$ where $c = \sup\{x \geqslant x_0\colon \phi(x) > 0\}$. It is also clear that

$$c = x_\infty := \sup\{x\colon (x, y) \in D(a)\} \text{ and } x_\infty > x_0$$

and furthermore that

$$\lim_{x \to x_\infty} \phi(x) = y_\infty := \inf\{y\colon (x, y) \in D(a)\}.$$

By the monotonicity of ϕ, the point $(x_0, \phi(x_0+))$ is the limit of the sequence of points $\{(x_0 + 1/n, \phi(x_0 + 1/n))\}_{n \in \mathbb{N}}$. So $(x_0, \phi(x_0+)) \in \overline{\Gamma(a)}$ and consequently $\phi(x_0+) = \phi(x_0)$, i.e. ϕ is also continuous (one sidedly) at $x = x_0$.

We now interchange the roles of x and y and define the function ψ on $[y_0, \infty)$ by $\psi(u) := \inf\{s \in \mathbb{R}, (s, u) \in \overline{\Gamma(a)}\}$. For exactly analogous reasons to above, ψ is continuous, non-negative, non-increasing and convex on $[y_0, \infty)$ and satisfies $\psi(y_0) = x_0$. Furthermore, the set $\{(x, y) \in \partial\Gamma(a)\colon y > y_0\}$ coincides with the graph $\{(\psi(y), y)\colon y_0 < y\}$, and $\{(x, y) \in D(a)\colon y > y_0\}$ coincides with $\{(\psi(y), y)\colon y_0 < y < c\}$, where $c = \sup\{y \geqslant y_0\colon \psi(y) > 0\}$, and so also $c = y_{-\infty} := \sup\{y\colon (x, y) \in D(a)\}$ and $y_{-\infty} > y_0$.

This completes the proof of Claim 1.3.

Claim 1.4. *The quantities x_∞ and $y_{-\infty}$ defined in* (13) *satisfy*

$$x_\infty = \lim_{t \to \infty} K(t) = \|a\|_{A_0^\sim} \quad \text{and} \quad y_{-\infty} = \lim_{t \to 0} \frac{K(t)}{t} = \|a\|_{A_1^\sim} \tag{14}$$

and the quantities

$$x_{-\infty} := \inf\{x \mid (x, y) \in D(a)\} \quad \text{and} \quad y_\infty := \inf\{y \mid (x, y) \in D(a)\} \tag{15}$$

satisfy

$$y_\infty = \lim_{t\to\infty} \frac{K(t)}{t} \quad \textit{and} \quad x_{-\infty} = \lim_{t\to 0} K(t). \tag{16}$$

Proof. We shall prove only the two formulae for the limits as t tends to ∞. The proofs of the remaining two formulae, when t tends to 0, are exactly analogous. Alternatively, they can be very quickly deduced from the formulae for $t\to\infty$ by using the "reversed" couple $\vec{B} = (B_0, B_1) = (A_1, A_0)$. For the same element $a \in A_0 + A_1 = B_0 + B_1$, the Gagliardo diagram $\Gamma_{\vec{B}}(a)$ with respect to $\vec{B}$ is of course equal to $\{(x,y)\in\mathbb{R}^2\colon (y,x)\in\Gamma(a)\}$. Since $K(t,a;\vec{B}) = tK(1/t)$, the rest of the argument is obvious.

For each $x_* \in (x_0, x_\infty)$, the convexity and monotonicity of ϕ guarantee that the left and right derivatives $\phi'_-(x_*)$ and $\phi'_+(x_*)$ of ϕ at x_* exist and satisfy $\phi'_-(x_*) \leqslant \phi'_+(x_*) \leqslant 0$. For each $t_* > 0$ for which $\phi'_-(x_*) \leqslant -1/t_* \leqslant \phi'_+(x_*)$, the line $x + t_* y = x_* + t_*\phi(x_*)$ passes through the point $(x_*, \phi(x_*))$ and, by the convexity of $\Gamma(a)$, does not intersect with the interior of $\Gamma(a)$. It follows that $K(t_*) = x_* + t_*\phi(x_*)$.

Suppose that $\{\xi_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence in (x_0, x_∞) which tends to x_∞. Then $\phi(\xi_n)$ tends to y_∞. For each n we choose t_n such that $\phi'_-(\xi_n) \leqslant -1/t_n \leqslant \phi'_+(\xi_n)$ and so

$$K(t_n) = \xi_n + t_n\phi(\xi_n). \tag{17}$$

Since $\phi'_+(\xi_n) \leqslant \phi'_-(\xi_{n+1})$, we also have that $t_n \leqslant t_{n+1}$ for each n.

Let us first consider the case when $x_\infty = \infty$. It is clear from (17) that $\lim_{t\to\infty} K(t) \geqslant \lim_{n\to\infty} K(t_n) = \infty$, and this establishes the first formula of (14). Furthermore, since $\{\phi(\xi_n)\}$ and $\{K(t_n)/t_n\}$ are both convergent sequences, so is $\{\xi_n/t_n\}$ and consequently $\lim_{n\to\infty} t_n = \infty$. For each fixed $x_* \in [x_0, x_\infty)$ and each $n \in \mathbb{N}$, the point $(x_*, \phi(x_*))$ lies on or above the line $x + t_n y = \xi_n + t_n\phi(\xi_n)$. So $x_*/t_n + \phi(x_*) \geqslant \xi_n/t_n + \phi(\xi_n)$. Taking limits, first as n tends to ∞, and then as x_* tends to x_∞, shows that $\lim_{n\to\infty} \xi_n/t_n = 0$. Consequently $\lim_{t\to\infty} K(t)/t = \lim_{x\to x_\infty} \phi(x) = y_\infty$ and we have established the first formula of (16).

It remains to consider the case when $x_\infty < \infty$. Here we must have $y_\infty = 0$, and, since $(x_\infty, 0) \in \overline{\Gamma(a)}$, this gives us that, for each $t > 0$, $K(t) \leqslant x_\infty + t\cdot 0 = x_\infty$. We immediately deduce that $\lim_{t\to\infty} K(t)/t = 0$, which is the first formula of (16), and also that

$$\lim_{t\to\infty} K(t) \leqslant x_\infty. \tag{18}$$

As in the previous case, (17) implies that $\lim_{t\to\infty} K(t) \geqslant \lim_{n\to\infty} K(t_n) \geqslant x_\infty$ (even though now we do not necessarily have $\lim_{n\to\infty} t_n = \infty$). This, together with (18), establishes the first formula of (14).

This completes the proof of Claim 1.4.

We will now construct a special finite or infinite sequence of points of $D(a)$ which will be denoted by $\{(x_n, y_n)\}_{v_{-\infty} < n < v_\infty}$. Our construction is related to others used for various purposes in several different papers, cf. e.g. [9, p. 227]; [10, p. 95] and also [11, Example 0, p. 56]. It is almost exactly the same construction as on p. 74 of [6], but here we will describe it somewhat more explicitly than in [6], using Claim 1.3 and the functions ϕ and ψ introduced in Claim 1.3. (As mentioned in [7], there is a minor misprint in [6], where the range of n is incorrectly stated to be $v_{-\infty} - 1 < n < v_\infty + 1$.) Note that the quantities $v_{\pm\infty}$ satisfy

$$-\infty \leqslant v_{-\infty} \leqslant -1 \quad \text{and} \quad 1 \leqslant v_\infty \leqslant \infty .$$

The construction depends on a fixed positive number r, which in [6] is chosen to equal $1+\sqrt{2}$. It is convenient that the authors of [6] had the foresight to write most of their proof for a general parameter $r>1$. In our variant here we actually want to choose $r=2$, but we too will present most of the proof for general r, again with a view to facilitating future improvements.

It is convenient to choose the point (x_0, y_0), defined as in the proof of Claim 1.3, as the first (and sometimes only) member of our sequence.

The members (x_n, y_n) of the sequence for $n \geqslant 1$ are constructed by the following recursive procedure: Suppose that $(x_{n-1}, y_{n-1}) \in D(a)$ has been chosen and that it satisfies $x_{n-1} \geqslant x_0$ and therefore also $y_{n-1} = \phi(x_{n-1})$. If

$$\text{either } rx_{n-1} \geqslant x_\infty \text{ or } \frac{1}{r} y_{n-1} \leqslant y_\infty , \tag{19}$$

then we set $v_\infty = n$ and terminate the procedure, i.e. we do not construct (x_m, y_m) for $m = n$ nor for any $m > n$. Otherwise the set

$$\left\{ x \in (x_{n-1}, x_\infty) \colon x \geqslant rx_{n-1}, \phi(x) \leqslant \tfrac{1}{r} \phi(x_{n-1}) \right\}$$

is non-empty and we choose x_n to be its infimum, and $y_n = \phi(x_n)$. Clearly $(x_n, y_n) \in D(a)$. By the continuity of ϕ we have

$$\text{either } \begin{cases} x_n = rx_{n-1}, \\ y_n \leqslant \dfrac{1}{r} y_{n-1} \end{cases} \quad \text{or } \begin{cases} x_n \geqslant rx_{n-1}, \\ y_n = \dfrac{1}{r} y_{n-1}. \end{cases} \tag{20}$$

If we can construct (x_n, y_n) in this way for every $n \in \mathbb{N}$, then we set $v_\infty = \infty$.

An exactly analogous procedure to that just described is used to recursively construct the members of the sequence (x_n, y_n) for $n \leqslant -1$: Suppose that $(x_{n+1}, y_{n+1}) \in D(a)$ has been chosen and that it satisfies $y_{n+1} \geqslant y_0$ and therefore also $x_{n+1} = \psi(y_{n+1})$. If

$$\text{either } \frac{1}{r} x_{n+1} \leqslant x_{-\infty} \text{ or } ry_{n+1} \geqslant y_{-\infty} , \tag{21}$$

then we set $v_{-\infty} = n$ and terminate the procedure, not constructing (x_m, y_m) for any $m \leqslant n$. Otherwise we choose y_n to be the infimum of the non-empty set $\{ y \in (y_{n+1}, y_{-\infty}) \colon y \geqslant ry_{n+1}, \psi(y) \leqslant \tfrac{1}{r} \psi(y_{n+1}) \}$ and $x_n = \psi(y_n)$. Again we have $(x_n, y_n) \in D(a)$ and the continuity of ψ ensures that

$$\text{either } \begin{cases} x_n = \dfrac{1}{r} x_{n+1}, \\ \dfrac{1}{r} y_n \geqslant y_{n+1} \end{cases} \quad \text{or } \begin{cases} x_n \leqslant \dfrac{1}{r} x_{n+1}, \\ \dfrac{1}{r} y_n = y_{n+1}. \end{cases} \tag{22}$$

If we can construct (x_n, y_n) in this way for every negative integer n, then we set $v_{-\infty} = -\infty$.

We now observe that, for each point (x_n, y_n) of the sequence $\{(x_n, y_n)\}_{v_{-\infty}<n<v_\infty}$ which we have just constructed, and for any choice of the arbitrarily small positive number ε appearing in the statement of Theorem 1.1, the point $((1+\varepsilon)x_n, (1+\varepsilon)y_n)$ is in $\Gamma(a)$. Thus we can assert, as on p. 75 of [6], that there exists a decomposition

$$a = a_n + a_n' \tag{23}$$

such that

$$||a_n||_{A_0} \leqslant (1+\varepsilon)x_n \quad \text{and} \quad ||a_n'||_{A_1} \leqslant (1+\varepsilon)y_n. \tag{24}$$

All steps of the proof up to this point can also be carried out (cf. [6]) when $\vec{A}$ is an arbitrary Banach couple. But now we will start using our hypothesis that $\vec{A}$ is a lattice couple. One consequence of this is that it suffices to consider the case where the element a chosen above is a function which takes only non-negative values. The general case can be deduced from this case by first finding a sequence $\{u_n\}$ with the specified properties for the function $|a|$ and then multiplying all elements of that sequence pointwise by $\operatorname{sgn}(a)$ to obtain an appropriate sequence for a itself.

Since a is now taken to be non-negative we can assume, without loss of generality, that the elements a_n and a_n' in (23) are also both non-negative functions. If they are not, we can simply replace them by two new functions $\tilde{a}_n$ and $\tilde{a}_n'$ which vanish wherever a vanishes and, on the support of a, are given by

$$\tilde{a}_n = \frac{|a_n|a}{|a_n| + |a_n'|} \quad \text{and} \quad \tilde{a}_n' = \frac{|a_n'|a}{|a_n| + |a_n'|}.$$

Clearly $\tilde{a}_n$ and $\tilde{a}_n'$ are both non-negative and satisfy $a = \tilde{a}_n + \tilde{a}_n'$. Furthermore (in view of (2)) they satisfy the norm estimates in (24).

We now define the sequence $\{u_n\}_{n\in\mathbb{Z}}$ using our non-negative functions a_n and a_n' and the formulae (2.8) on p. 75 of [6]. Then, proceeding exactly as in [6], we first obtain that $\sum_{n=-\infty}^{\infty} u_n = a$, where the series converges in $A_0 + A_1$ norm, and also estimates (2.9) and (2.10) of [6], which amount to saying that

$$||u_n||_{A_0} \leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)x_n \quad \text{for all } n < v_\infty \tag{25}$$

and

$$||u_n||_{A_1} \leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)y_{n-1} \quad \text{for all } n > v_{-\infty}+1. \tag{26}$$

We do not follow the next steps in [6] since we do not need the more elaborate estimates of pp. 76–77 for the sums $\sum_{n=-\infty}^{\infty} \min\{||u_n||_{A_0}, t||u_n||_{A_1}\}$ and $\sum_{n=-\infty}^{\infty} \min\{||u_n||_{A_0^\sim}, t||u_n||_{A_1^\sim}\}$. (Instead, later, we will obtain and use sharpened versions of those estimates.) At this stage it suffices to show something rather simpler, namely that

$$\sum_{n=-\infty}^{\infty} ||u_n||_{A_0+A_1} < \infty. \tag{27}$$

Conditions (20) and (22) imply that the numbers x_n and y_n in our construction satisfy

$$x_m \leqslant r^{m-n} x_n \quad \text{and} \quad y_n \leqslant r^{m-n} y_m \tag{28}$$

for any integers m and n satisfying $v_{-\infty} < m \leqslant n < v_\infty$. Since $u_n = 0$ for $n \leqslant v_{-\infty}$ (if $v_{-\infty} > -\infty$) and also for $n > v_\infty$ (if $v_\infty < \infty$), we obtain from (25), (26) and (28) that

$$\sum_{n=-\infty}^{0} ||u_n||_{A_0} = \sum_{n=v_{-\infty}+1}^{0} ||u_n||_{A_0} \leqslant \sum_{n=-\infty}^{0} (1+\varepsilon)\left(1+\frac{1}{r}\right) r^n x_0 < \infty$$

and, analogously,

$$\begin{aligned}\sum_{n=1}^{\infty} ||u_n||_{A_1} &= \sum_{n=1}^{v_\infty} ||u_n||_{A_1} \leqslant \sum_{n=1}^{v_\infty} (1+\varepsilon)\left(1+\frac{1}{r}\right) y_{n-1} \\ &= \sum_{n=0}^{v_\infty - 1} (1+\varepsilon)\left(1+\frac{1}{r}\right) y_n \\ &\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right) \sum_{n=0}^{v_\infty - 1} r^{-n} y_0 < \infty.\end{aligned}$$

These two estimates imply (27).

Let $(\Omega, \mathscr{S}, \mu)$ be the underlying measure space for the Banach lattices A_0 and A_1. We introduce the measurable sets $E_n = \{\omega \in \Omega : u_n(\omega) > 0\}$ and the non-negative functions $g_n = u_n \chi_{E_n}$ for each $n \in \mathbb{Z}$.

It follows from (27) and Proposition 0.2 that the series $\sum_{n=-\infty}^{\infty} u_n$ converges to a pointwise a.e., as well as in $A_0 + A_1$ norm. Since $||g_n||_{A_0+A_1} \leqslant ||u_n||_{A_0+A_1}$ for each n, we deduce, again using Proposition 0.2, that the series $\sum_{n=-\infty}^{\infty} g_n$ also converges, pointwise a.e. and also in $A_0 + A_1$ norm, to a function $g \in A_0 + A_1$. Since $u_n \leqslant g_n$ a.e., this implies that $a(\omega) \leqslant g(\omega)$ for a.e. $\omega \in \Omega$.

Now we need to use the fact that a is non-negative a second time. It implies that $a(\omega) = 0$ for a.e. ω in the set where $g(\omega) = 0$. Furthermore, the function $\phi : \Omega \to \mathbb{R}$, defined by

$$\phi(\omega) = \begin{cases} 0 & \text{if } g(\omega) = 0 \\ a(\omega)/g(\omega) & \text{if } g(\omega) \neq 0, \end{cases}$$

satisfies $0 \leqslant \phi(\omega) \leqslant 1$ for a.e. $\omega \in \Omega$. We can now define a new sequence $\{f_n\}_{n \in \mathbb{Z}}$ which will be our "improvement" of the sequence $\{u_n\}_{n \in \mathbb{Z}}$. It is given by

$$f_n = g_n \phi \quad \text{for each } n \in \mathbb{Z}.$$

We will see that Theorem 1.1 follows when we replace each u_n by f_n. Obviously $\sum_{n=-\infty}^{\infty} f_n = g\phi = a$ a.e., and as well as converging pointwise a.e., this series also converges to a in $A_0 + A_1$ norm. (We have used Proposition 0.2 once more here.) So to complete the proof it will suffice to show that

$$\sum_{n=-\infty}^{\infty} \min\{||f_n||_{A_0^{\sim}}, t||f_n||_{A_1^{\sim}}\} \leqslant 4(1+\varepsilon) K(t, a; \vec{A}) \quad \text{for all } t > 0. \tag{29}$$

Thus we turn to estimating $||f_n||_{A_0}$ and $||f_n||_{A_1}$. If $v_{-\infty}+1<n<v_\infty$ then $0\leqslant a_n(\omega)-a_{n-1}(\omega)$ for all $\omega\in E_n$. But also, since a_{n-1} is non-negative everywhere, we have $a_n(\omega)-a_{n-1}(\omega)\leqslant a_n(\omega)$. Consequently,

$$\begin{aligned}||f_n||_{A_0} &\leqslant ||g_n||_{A_0}=||(a_n-a_{n-1})\chi_{E_n}||_{A_0}\leqslant ||a_n\chi_{E_n}||_{A_0}\\ &\leqslant (1+\varepsilon)x_n.\end{aligned}\tag{30}$$

We observe that (30) obviously holds also for $n=v_{-\infty}+1$ if $v_{-\infty}$ is finite. Analogous estimates hold for $||f_n||_{A_1}$: If $v_{-\infty}+1<n<v_\infty$, then $0\leqslant a'_{n-1}-a'_n\leqslant a'_{n-1}$ on E_n. Consequently

$$||f_n||_{A_1}\leqslant ||a'_{n-1}||_{A_1}\leqslant (1+\varepsilon)y_{n-1},\tag{31}$$

and again the same estimate obviously holds for $n=v_\infty$, if v_∞ is finite.

If $v_\infty<\infty$ then, for $n=v_\infty$, we will sometimes need the following substitute for (30): Since $0\leqslant a-a_{v_\infty-1}$ on E_n and $a_{v_\infty-1}$ is a non-negative function, then $0\leqslant f_n\leqslant g_n\leqslant a$ and so, by (14),

$$||f_{v_\infty}||_{A_0^\sim}\leqslant ||a||_{A_0^\sim}=x_\infty.\tag{32}$$

Similarly, if $v_{-\infty}>-\infty$ then, for $n=v_{-\infty}+1$, instead of (31) we have, by an exactly analogous argument to the one establishing (32) including the use of the second formula in (14), that

$$||f_{v_{-\infty}+1}||_{A_1^\sim}\leqslant ||a||_{A_1^\sim}=y_{-\infty}.\tag{33}$$

Observe that the two estimates (30) and (31) are improved analogues of the estimates (25) and (26) (i.e. (2.9) and (2.10) on p. 75 of [6]) for $||u_n||_{A_0}$ and $||u_n||_{A_1}$ respectively. Here we have been able to remove the factor $(1+1/r)$ which appeared in both of those estimates.

It will be convenient at this stage to express the set $\partial\Gamma(a)$ as the union of a special sequence of its subsets. For each integer n in the range $1\leqslant n<v_\infty$ we define $D_n=\{(x,y)\in\partial\Gamma(a)\colon x_{n-1}\leqslant x\leqslant x_n, y\leqslant y_0\}$ This means (see Claim 1.3) that $D_n=\{(x,\phi(x))\colon x_{n-1}\leqslant x\leqslant x_n\}$. Analogously, for each integer n in the range $v_{-\infty}+1<n\leqslant 0$, we define $D_n=\{(x,y)\in\partial\Gamma(a)\colon y_n\leqslant y\leqslant y_{n-1}, x\leqslant x_0\}$ which means that $D_n=\{(\psi(y),y)\colon y_n\leqslant y\leqslant y_{n-1}\}$.

In view of the monotonicity of the functions ϕ and ψ, we have

$$D_n\subset\{(x,y)\in D(a)\colon x_{n-1}\leqslant x\leqslant x_n, y_n\leqslant y\leqslant y_{n-1}\}\tag{34}$$

for all n in the range $v_{-\infty}+1<n<v_\infty$.

Using (28) with $m=0$ and $n>0$ arbitrarily large, and then (11) and (12), we obtain the implications

$$v_\infty=\infty\Rightarrow x_\infty=\infty\Rightarrow\partial\Gamma(a)_+=D(a)_+=\bigcup_{n\geqslant 1}D_n.\tag{35}$$

An analogous argument (with $n=0$ and m tending to $-\infty$) shows that

$$v_{-\infty}=-\infty\Rightarrow y_{-\infty}=\infty\Rightarrow\partial\Gamma(a)_-=D(a)_-=\bigcup_{n\leqslant 0}D_n.\tag{36}$$

We need two more sets, D_∞ and $D_{-\infty}$. We set $D_\infty = \{(x,y)\in\partial\Gamma(a):\ x\geqslant x_{v_\infty-1}\}$ if $v_\infty<\infty$, and $D_\infty=\emptyset$ if $v_\infty=\infty$. Similarly $D_{-\infty}=\{(x,y)\in\partial\Gamma(a):\ y\geqslant y_{v_{-\infty}+1}\}$ if $v_{-\infty}>-\infty$, and otherwise it is empty. In all cases, whether or not v_∞ and/or $v_{-\infty}$ are infinite, we have, using (35), (36) and (11), that

$$\partial\Gamma(a)=D_{-\infty}\cup\Delta\cup D_\infty,\quad \text{where } \Delta=\bigcup_{v_{-\infty}+1<n<v_\infty} D_n. \tag{37}$$

Finally, we are ready to estimate $\sum_{n=-\infty}^{\infty}\min\{\|f_n\|_{A_0^{\sim}},t\|f_n\|_{A_1^{\sim}}\}$. Recalling property (9), for each fixed $t>0$, we let Λ_t denote the intersection of the line $x+ty=K(t)$ with $\partial\Gamma(a)$. At least one of the three sets $\Delta\cap\Lambda_t$, $D_\infty\cap\Lambda_t$ and $D_{-\infty}\cap\Lambda_t$ is non-empty, and our proof will consider each of these three possibilities as separate cases. (These are, in fact, the same three cases which are mentioned in the last paragraph on [6, p. 75].)

Case 1 is when $\Delta\cap\Lambda_t\neq\emptyset$. In this case we let (x^*,y^*) denote a point of Λ_t and n^* denote an integer in the interval $(v_{-\infty}+1,v_\infty)$ such that $(x^*,y^*)\in D_{n^*}$. By (34) we have that $x_{n^*-1}\leqslant x^*\leqslant x_{n^*}$ and also $y_{n^*-1}\geqslant y^*\geqslant y_{n^*}$. This enables us to use almost exactly the same estimates as on p. 76 of [6]. The only changes are that u_n is of course replaced by f_n, and, since we have now replaced (25) and (26) by (30) and (31), the factor $(1+1/r)$ does not appear. We thus obtain that

$$\sum_{n=-\infty}^{\infty}\min\{\|f_n\|_{A_0},t\|f_n\|_{A_1}\}\leqslant(1+\varepsilon)\left(\left(1-\frac{1}{r}\right)^{-1}+r\right)K(t). \tag{38}$$

We now consider Case 2, which is when $D_\infty\cap\Lambda_t\neq\emptyset$. Here we have $v_\infty<\infty$ and $x^*\geqslant x_{v_\infty-1}$ for some point $(x^*,y^*)\in\Lambda_t$. So, using (5), we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\min\{\|f_n\|_{A_0^{\sim}},t\|f_n\|_{A_1^{\sim}}\}&=\sum_{n=-\infty}^{v_\infty}\min\{\|f_n\|_{A_0^{\sim}},t\|f_n\|_{A_1^{\sim}}\}\\&\leqslant\min\{\|f_{v_\infty}\|_{A_0^{\sim}},t\|f_{v_\infty}\|_{A_1^{\sim}}\}+\sum_{n=-\infty}^{v_\infty-1}\|f_n\|_{A_0}.\end{aligned}$$

The second term here is estimated in a similar way to that used for Case 1. More specifically, by (30) and (28), we have

$$\begin{aligned}\sum_{n=-\infty}^{v_\infty-1}\|f_n\|_{A_0}&=\sum_{n=v_{-\infty}+1}^{v_\infty-1}\|f_n\|_{A_0}\leqslant(1+\varepsilon)\sum_{n=v_{-\infty}+1}^{v_\infty-1}x_n\\&\leqslant(1+\varepsilon)\sum_{n=v_{-\infty}+1}^{v_\infty-1}r^{v_\infty-1-n}x_{v_\infty-1}\leqslant(1+\varepsilon)\left(1-\frac{1}{r}\right)^{-1}x_*\\&\leqslant(1+\varepsilon)\left(1-\frac{1}{r}\right)^{-1}K(t).\end{aligned}$$

Now we consider the first term: The fact that $v_\infty<\infty$ means that either $rx_{v_\infty-1}\geqslant x_\infty$ (call this "Subcase 2a") or $\frac{1}{r}y_{v_\infty-1}\leqslant y_\infty$ ("Subcase 2b"). In Subcase 2a we have, using (32), that

$$\min\{\|f_{v_\infty}\|_{A_0^\sim}, t\|f_{v_\infty}\|_{A_1^\sim}\} \leqslant \|f_{v_\infty}\|_{A_0^\sim} \leqslant x_\infty \leqslant rx^* \leqslant rK(t).$$

In Subcase 2b we use (5), (31), (16) and the fact that $K(t)/t$ is a non-increasing function, to obtain that

$$\min\{\|f_{v_\infty}\|_{A_0^\sim}, t\|f_{v_\infty}\|_{A_1^\sim}\} \leqslant t\|f_{v_\infty}\|_{A_1} \leqslant (1+\varepsilon)ty_{v_\infty - 1} \leqslant (1+\varepsilon)try_\infty \leqslant (1+\varepsilon)rK(t).$$

Combining the preceding estimates gives us that, in both subcases,

$$\sum_{n=-\infty}^{\infty} \min\{\|f_n\|_{A_0^\sim}, t\|f_n\|_{A_1^\sim}\} \leqslant (1+\varepsilon)\left(r + \left(1 - \frac{1}{r}\right)^{-1}\right)K(t). \tag{39}$$

The final case which we have to consider, Case 3, is when $D_{-\infty} \cap \Lambda_t \neq \emptyset$, so that $v_{-\infty} > -\infty$ and $y^* \geqslant y_{v_{-\infty}+1}$ for some point $(x^*, y^*) \in \Lambda_t$. Here again there are two subcases, depending on whether $\frac{1}{r}x_{v_{-\infty}+1} \leqslant x_{-\infty}$ or $ry_{v_{-\infty}+1} \geqslant y_{-\infty}$. We leave it to the reader to provide the details of the arguments for this case. They are completely analogous to those of Case 2 (or they could, perhaps somewhat tediously, be deduced from Case 2 applied to the "reversed" couple $\vec{B}$ mentioned at the beginning of the proof of Claim 1.4). These arguments again lead to estimate (39). Of course (5) and (38) also give us (39) in Case 1.

It now remains only to recall that in fact we intended r to be equal to 2. Substituting this optimal choice of r converts (39) into (29) and completes the proof of Theorem 1.1. As already explained, this also completes the proof of our main result, the estimate (4) for all lattice couples.

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